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CRITICAL GRAPHS, MATCHINGS AND TOURS OR A HIERARCHY OF RELAXATIONS FOR THE TRAVELLING SALESMAN PROBLEM

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Dedicated to Tibor Gallai on his seventieth birthday

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A *(perfect) 2-matching* in a graph $G=(V, E)$ is an assignment of an integer 0, 1 or 2 to each edge of the graph in such a way that the sum over the edges incident with each node is at most (exactly) two. The incidence vector of a Hamiltonian cycle, if one exists in G, is an example of a perfect 2-matching. For k satisfying $1 \leq k \leq |V|$, we let P_k denote the problem of finding a perfect 2-matching of G such that any cycle in the solution contains more than k edges. We call such a matching a *perfect P_k-matching*. Then for $k<l$, the problem P_k is a relaxation of P_l. Moreover if $|V|$ is odd, then $P_{|V|-2}$ is simply the problem of determining whether or not G is Hamiltonian. A graph is *P_k-critical* if it has no perfect P_k-matching but whenever any node is deleted the resulting graph does have one. If $k=|V|$, then a graph $G=(V, E)$ is P_k-critical if and only if it is *hypomatchable* (the graph has an odd number of nodes and whatever node is deleted the resulting graph has a perfect matching). We prove the following results:

1. If a graph is P_k-critical, then it is also P_l-critical for all larger l. In particular, for all k, P_k-critical graphs are hypomatchable.
2. A graph $G=(V, E)$ has a perfect P_k-matching if and only if for any $X \subseteq V$ the number of P_k-critical components in $G[V-X]$ is not greater than $|X|$.
3. The problem P_k can be solved in polynomial time provided we can recognize P_k-critical graphs in polynomial time. In addition, we describe a procedure for recognizing P_k-critical graphs which is polynomial in the size of the graph and exponential in k.

1. Introduction

Let P be any property possessed by some graphs. We say that a graph $G=(V, E)$ is *P-critical* if G does not have property P, but $G[V-\{v\}]$ does have the property for every vertex $v \in V$. (Here $G[S]$ denotes the subgraph of G induced by the vertex set $S \subseteq V$.)

There are several examples of properties P for which the P-critical graphs are quite well-known. A *Hamiltonian cycle* of G is a cycle passing through each node exactly once. If P is the property "G possesses a Hamiltonian cycle", then the P-critical

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graphs are the *hypohamiltonian* graphs, the smallest example of which is the Petersen graph. A *matching* of a graph is a set of nonadjacent edges. A matching is *perfect* if each vertex of the graph is incident with one edge of the matching. If P is the property "G possesses a perfect matching", then the P-critical graphs are the *hypomatchable* graphs (sometimes called *factor-critical* graphs).

Two questions arise immediately for a given property P:

1. Are the P-critical graphs of any intrinsic interest?
2. Can we characterize them?

Both questions have been answered positively for hypomatchable graphs. The answers will serve as an introduction to this paper.

First we introduce some notation. Consider a graph $G=(V, E)$ and a vertex set $S\subseteqq V$. We let $\gamma(S)$ denote the set of all edges of G with both ends in S, and let $\delta(S)$ denote the set of edges of G that have exactly one end in S. We abbreviate $\delta(\{v\})$ by $\delta(v)$ for $v\in V$. Let $\mathbf{R}^J$ be the set of all real vectors $x=(x_j: j\in J)$ where J is some finite index set. For any $x\in\mathbf{R}^J$ and any $K\subseteqq J$ we let $x(K)$ denote $\sum_{j\in K} x_j$. Any subset $F\subseteqq E$ of the edges of G is identified with a 0,1 vector $x=(x_e: e\in E)$ where $x_e=1$ if $e\in F$ and 0 otherwise.

Edmonds' matching polyhedron theorem is closely tied to hypomatchable graphs, as is a strengthening of Tutte's characterization of those graphs that have a perfect matching. A graph with no cutnode is called *nonseparable*.

Theorem 1.1. (Edmonds [6]) *The convex hull of the matchings of G is*

$$\{x\in\mathbf{R}^E:\ x_e \geqq 0 \quad \text{for all} \quad e\in E, \tag{1.1}$$

$$x(\delta(v)) \leqq 1 \quad \text{for all} \quad v\in V, \tag{1.2}$$

$$x(\gamma(S)) \leqq (|S|-1)/2 \quad \text{for all} \quad S\subseteqq V \tag{1.3}$$

such that $G[S]$ *is nonseparable and hypomatchable*$\}$.

Moreover, Pulleyblank and Edmonds [12] showed that all the inequalities (1.3) are facets inducing and hence are essential.

Theorem 1.1 was proved by Edmonds by means of an algorithm which, for any $c\in\mathbf{R}^E$, produces an optimum solution to the linear program: maximize $\{cx$: x satisfies (1.1)—(1.3)$\}$ which is integer valued. He also gave [5] a specialization of this algorithm for finding a maximum cardinality matching. This algorithm proves the following:

Theorem 1.2. *A graph G has a perfect matching if and only if, for every* $S\subseteqq V$, *the number of hypomatchable components of* $G[V-S]$ *is at most* $|S|$.

Since a hypomatchable graph always has an odd number of vertices, the necessity of this condition is easy. The sufficiency is a strengthening of Tutte's theorem [13], which is the same as Theorem 1.2 but with "hypomatchable" replaced by "odd". A nonalgorithmic proof of Theorem 1.2 was given by Anderson [1].

The central part played by hypomatchable graphs in Theorems 1.1 and 1.2 gives an affirmative answer to the first question asked in the introduction. To answer the second question we mention two constructions of the family of hypomatchable graphs. The first one is in terms of Edmonds' matching algorithm: the hypomatchable graphs are precisely those that are shrinkable.

Let $G=(V, E)$ be a graph and $S \subseteq V$. We define $G \times S$, the *graph obtained from G by shrinking S*, to be the graph obtained from G by deleting all edges of $\gamma(S)$ and combining all the nodes of S to from a *pseudonode* which we call S.

Theorem 1.3. (Shrinkability) (Pulleyblank, Edmonds [12]) *A graph G is hypomatchable if and only if there exists a sequence $G_0, G_1, \ldots, G_p$ of graphs such that*
a) $G_0=G$
b) G_p *consists of a single pseudonode*
c) *for each* $i=1, \ldots, p$, $G_i=G_{i-1} \times S_i$, *where* S_i *is the vertex set of an odd cycle of* G_{i-1}.

It is interesting to note that the above shrinking sequence for G can be consstructed in such a way that for each G_i, $i=1, \ldots, p$, all nodes but one are nodes of G. Thus each G_i for $i=1, 2, \ldots, p$ will contain exactly one pseudonode. This requires that each S_i contains the pseudonode S_{i-1} for $i=2, \ldots, p$. Such a shrinking corresponds to what is called an *ear decomposition* of G.

Theorem 1.4. (Ear decomposition) (Lovász [7]) *A graph* $G=(V, E)$ *is hypomatchable if and only if there exists a sequence of vertex sets* V_i, $0 \leqq i \leqq p$, *such that*
a) V_0 *is the vertex set of an odd cycle of G,*
b) $V_{i-1} \subset V_i$ *for* $1 \leqq i \leqq p$. *Furthermore* $V_i - V_{i-1}$ *has even cardinality and is the vertex set of a simple path* P_i *of G both of whose endpoints are adjacent to* V_{i-1},
c) $V_p = V$.

In Section 3 we will prove a further specialization: If G is nonseparable and hypomatchable, then an ear decomposition can be found such that $G[V_i]$ is nonseparable for all $i=0, 1, 2, \ldots, p$.

2. A family of relaxations of the Hamilton cycle problem

We denote the properties that a graph has a Hamilton cycle and a perfect matching by H and M respectively.

A *2-matching* of a graph G is an assignment of the integers 0, 1, 2 to the edges of G such that for each vertex the sum of the integers on the incident edges is at most 2. If the sum equals 2 for every vertex, then we say that the 2-matching is *perfect*. We let P_1 be the property that a graph has a perfect 2-matching.

Clearly, the incidence vector of every Hamilton cycle is a 2-matching, so P_1 is a relaxation of H. Assigning the value 2 to the edges of a matching yields a 2-matching, so P_1 is also a relaxation of M. Stronger relaxations can be obtained.

The edges that are assigned the value 1 in a perfect 2-matching form disjoint cycles, called its *polygons*. A perfect 2-matching is called a *perfect P_k-matching* if each of its polygons contains more than k edges. We denote by P_k the property that a graph has a perfect P_k-matching, for $k \geqq 2$. Obviously $P_2 = P_1$ and P_k is a relaxation of P_h for $h > k \geqq 2$. The following result is well-known.

Lemma 2.1. *If G has a perfect 2-matching x, then it has one with the same set of odd polygons as x and no even polygon.*

Proof. Remove each even polygon of x by assigning 2 to every other of its edges and zero to the rest. ▌

As a consequence, $P_{2i-1} \equiv P_{2i}$ for $i \geqq 2$. Finally let n be the number of vertices of the graph G. When n is odd $P_{n-2} (\equiv P_{n-1}) \equiv H$ since every perfect 2-matching must contain at least one odd polygon and the only polygon allowed by P_{n-2} or P_{n-1} is a Hamilton cycle. Note also that for n odd no graph possesses P_k for $k \geqq n$. On the other hand, when n is even, $P_k \equiv M$ for all $k \geqq n-1$.

Thus the full hierarchy of relaxations of H, starting with P_1, is:

$$P_1 \equiv P_2 \to P_3 \equiv \ldots \to P_{2i-1} \equiv P_{2i} \to \ldots P_{n-1} \equiv \begin{matrix} H \text{ if } n \text{ is odd} \\ P_n \equiv M \to H \text{ if } n \text{ is even,} \end{matrix}$$

where "$\to$" denotes "is a relaxation of".

For each of these preperties a family of critical graphs is defined according to the definition of Section 1. G is *P_k-critical* if and only if G does not have a perfect P_k-matching but $G[V-\{v\}]$ does, for every $v \in V$.

Theorem 2.2. *If G is P_k-critical, then G is hypomatchable.*

We prove this theorem in Section 3, as well as other properties of P_k-critical graphs.

One consequence of Theorem 2.2 is that every P_k-critical graph has an odd number of vertices. Another consequence is that the P_k-critical graphs themselves form a hierarchy: if a graph is P_k-critical, then it is P_h-critical for every $h \geqq k$ since it is hypomatchable and any perfect P_h-matching is also a perfect P_k-matching. The hypomatchable graphs with n vertices are precisely the P_n-critical graphs with n vertices for n odd, $n \geqq 3$. (See Figure 1.)

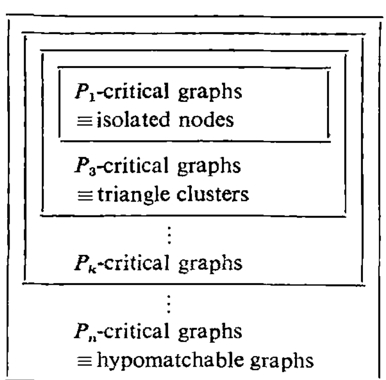

Fig. 1. Hierarchy of P_k-critical graphs with n (odd) nodes

Now we summarize some of the known results about the relaxations P_k and the P_k-critical graphs.

P_1 is the property of posessing a perfect 2-matching. Edmonds (see [10]) showed how to check in polynomial time whether a graph has property P_1. It uses a simplified version of the b-matching algorithm that performs no shrinking. (It is also possible to reduce this problem to bipartite matching or, equivalently, to finding the maximum flow in a network.) Perhaps one reason for the simplicity of this problem is that the only P_1-critical graph consists of a single isolated node (Pulleyblank [11]). A general theorem of Tutte [14] specializes to the following for P_1.

Theorem 2.3. *A graph G has a perfect 2-matching if and only if for every $S \subseteqq V$ the number of connected components of $G[V-S]$ which are isolated nodes is not greater than $|S|$.*

P_3 is the property that a graph has a perfect triangle-free 2-matching. Finding whether a graph has property P_3 was solved in [3]. The algorithm is based on the concept of a *triangle cluster,* that is, a connected graph whose every block is a triangle. The triangle clusters are precisely the P_3-critical graphs. In [3], the following theorem was proved.

Theorem 2.4. *A graph G has a perfect triangle-free 2-matching if and only if, for every $S \subseteqq V$, the number of connected components of $G[V-S]$ which are triangle clusters is not greater than $|S|$.*

As we have already seen in Section 1, the property that a graph has a perfect matching (property M) is treated in very much the same way as P_1 and P_3: Edmonds [5] gave a polynomial algorithm, the critical graphs are characterized via the ear decomposition and Tutte's theorem holds, see Theorem 1.2. Recall also that $M \equiv P_n$ for the graphs with at most n vertices.

This unity of results for properties P_1, P_3 and P_n (for graphs with at most n vertices) suggests that similar results may also hold for every P_k, $k \geqq 1$.

In Section 4 we show that this is indeed the case. We present an algorithm for recognizing whether a graph has the property P_k, and when it does, the algorithm finds a perfect P_k-matching. This algorithm is based upon an oracle for recognizing P_k-critical graphs. As a by-product of the algorithm we get:

Theorem 2.5. *A graph G has a perfect P_k-matching if and only if, for every $S \subseteqq V$, the number of P_k-critical components of $G[V-S]$ is not greater than $|S|$.*

In Section 5 we show how to recognize P_k-critical graphs in polynomial time (as a function of the graph size) for any fixed value of k. We discuss the case of P_5-critical graphs in some depth.

In Section 6, however, we remark that the polyhedral theorems known for P_k-matchings (not necessarily perfect) when $k=1$, 3 or $|V|$ (see [2], [6]) do not generalize in a simple way to $5 \leqq k < |V|$. As a consequence, the unity of results found in this paper stops with the maximum cardinality P_k-matching problem (see Section 4); the problem of finding a maximum weight P_k-matching in a graph with weighted edges seems intrinsicly harder for $5 \leqq k < |V|$ than for the other values of k.

3. Properties of hypomatchable and P_k-critical graphs

First we prove Theorem 2.2 which was stated in Section 2.

Theorem 2.2. *If G is P_k-critical, then G is hypomatchable.*

Proof. Let V be the vertex set of G and let $v \in V$. Since G is P_k-critical, $G[V-\{v\}]$ has a perfect P_k-matching. Let x be such a perfect P_k-matching having no even polygon (see Lemma 2.1) and the smallest possible number of odd polygons.

In G, x is a 2-matching deficient at vertex v. Define the 0, 1 vector y as $y_e=1$ if $x_e=2$ and $y_e=0$ otherwise. Then y is the incidence vector of a matching of G deficient at vertex v and possibly some other vertices. Apply Edmonds' matching algorithm to attempt to augment y. An alternating tree rooted at v is grown relative to y. When an edge joins two even nodes of the alternating tree, the blossom is shrunk.

Note that none of the shrunk nodes can have a perfect P_k-matching (otherwise an augmentation of x would be possible, yielding a perfect P_k-matching of G— a contradiction). However, the shrunk nodes in Edmonds' matching algorithm are hypomatchable. Therefore they are P_k-critical.

Note also that the odd polygons of x can only be adjacent to odd nodes of the alternating tree (otherwise an augmentation of x would again be possible — a contradiction).

So Edmonds' algorithm keeps growing the alternating tree without finding an augmentation. When the tree is completely grown either it still contains some odd nodes or it reduces to a single shrunk node, namely the root.

In the first case, let u be an odd node of the alternating tree. We use a classical argument in matching theory to show that $G[V-\{u\}]$ does not have a perfect P_k-matching. All the even nodes of the alternating tree are P_k-critical components, say $G[S_i]$ for $i=1, \ldots, s$. Thus any perfect P_k-matching x^0 of $G[V-\{u\}]$ must satisfy $x^0(\delta(S_i)) \geqq 2$ for $i=1, \ldots, s$. But even nodes are only adjacent to odd nodes of the alternating tree and the number of even nodes exceeds that of the odd nodes, even before node u is removed. So no perfect P_k-matching of $G[V-\{u\}]$ can exist. This contradicts the fact that G is P_k-critical.

Consequently, the second case must hold when the tree is completely grown, i.e. the tree must reduce to a single shrunk node. If x had an odd polygon, G would be disconnected since this odd polygon is not adjacent to the shrunk root of the tree, a contradiction.

So x contains no odd polygon, i.e. $y=x/2$ is a perfect matching of $G[V-\{v\}]$. This being true for every $v \in V$, G is hypomatchable. □

For any hypomatchable graph, we can ask the question: What is the smallest value k for which G is P_k-critical? The following result shows that this is equivalent to the question: What is the size of the largest odd polygon C of G such that $G[V-V(C)]$ has a perfect matching? (Here $V(C)$ denotes the set of vertices of the polygon C.)

Theorem 3.1. *Let G be a hypomatchable graph and let x be a perfect P_k-matching of G containing a minimum number of polygons. Then x contains exactly one polygon.*

Proof. By Lemma 2.1, x contains no even polygons. Suppose that x contained more than one polygon. The number of these polygons is odd, and hence at least three, since G has an odd number of vertices. We define a matching y of G as follows:

$$y_j = \begin{cases} 0 & \text{if } x_j = 0 \\ 1 & \text{if } x_j = 2 \\ 0 \text{ or } 1 & \text{if } x_j = 1, \text{ so that } y \text{ will be a maximum matching on the odd polygon containing } j. \end{cases}$$

Then every polygon C of x will contain exactly one vertex $v(C)$ not saturated by y, and these are the only vertices not saturated by y. If we delete from G one such vertex $v(C)$, the resulting graph has a perfect matching, that is, y is not maximal. So, by Berge's theorem, there exists an augmenting path relative to y, joining $v(C_1)$ to $v(C_2)$ where C_1 and C_2 are some polygons of x. This augmenting path must contain a path P whose end vertices belong to different polygons of x, say C_1' and C_2', and such that no other vertex of P belongs to a polygon of x. Note that the first and last edge of P are assigned the value 0 in x, so P is an augmenting path. By changing x on the polygons C_1' and C_2' and applying the augmentation along P, we can obtain a new P_k-matching x' with a smaller number of polygons, a contradiction. ∎

In fact, Theorem 3.1 relates quite nicely to the structure theorems for hypomatchable graphs as will be shown in the next theorem. First we desribe how to construct the ear decomposition defined in Theorem 1.4. (See also Lovász [8]) Given a hypomatchable graph G, the construction can be done relative to any near-perfect matching x. Assume that x is deficient at node v. Then each V_i of the ear decomposition will contain v and $G[V_i]$ will be near-perfectly matched by x, for every $i=0, \ldots, p$.

The Construction. Let $V_{-1}=\{v\}$.

For $i=0, 1, \ldots, p$, choose a node $u_i \in V-V_{i-1}$ which is adjacent to some node of V_{i-1}. (This is always possible since G is hypomatchable and thus connected.) Now consider a near-perfect matching x^i deficient at node u_i. Starting from u_i and using edges alternately in x and x^i we follow a path which eventually reaches the vertex set V_{i-1}, say from the vertex $v_i \in V-V_{i-1}$. Consider the portion P_i of this alternating path between u_i and v_i. The first edge in P_i belongs to x and, since $x(\delta(V_{i-1}))=0$, its last edge also belongs to x. Therefore P_i contains an even number of vertices. Since its two endpoints are adjacent to V_{i-1}, it satisfies the condition (b) of Theorem 1.4. Consequently we can define V_i-V_{i-1} to be the vertex set of P_i and perform the next iteration of the construction.

The path P_i is called an *ear*. Two vertices of V_{i-1}, adjacent to u_i and v_i respectively, are distinguished and called the *attachments* of the ear P_i. Note that these two vertices need not be distinct.

Remark 3.2. At each step of the construction, the choice of node $u_i \in V-V_{i-1}$ is arbitrary as long as u_i is adjacent to V_{i-1}.

Theorem 3.3. *Let P be an odd polygon of a hypomatchable graph G. Then the following statements are equivalent.*

(i) *There exists a perfect 2-matching of G for which P is the only odd polygon,*

(ii) *P can be the odd cycle of an ear decomposition of G,*

(iii) *P can be the first polygon shrunk in a shrinking of G.*

Proof. (i)$\Rightarrow$(ii): The perfect 2-matching y of which P is the only polygon can be used to define two matchings, say x and x^0, where $x_e = x_e^0 = y_e/2$ if $e \notin P$ and each of the matchings near-perfectly matches P, their respective deficient nodes being two adjacent nodes of P, say v and u_0. Then, by the construction of the ear decomposition introduced above, V_0 will be the vertex set of P.
(ii)$\Rightarrow$(iii): This follows from the remark made in Section 1 that an ear decomposition is a special case of shrinking.
(iii)$\Rightarrow$(i): After shrinking P, the graph $G \times P$ is hypomatchable and therefore it can be near-perfectly matched so that its pseudo-node is deficient. The 2-matching required in (i) follows. □

In Section 5 we will use the following lemmas.

Lemma 3.4. *Let G be a hypomatchable graph. Then, for any pair of vertices u and v, there exists an even length path joining them such that its deletion leaves a graph with a perfect matching.*

Proof. Let x^u and x^v be perfect matchings of $G[V-\{u\}]$ and $G[V-\{v\}]$ respectively. Define $x = x^u + x^v$. Then u and v are joined by an even path all of whose edges are assigned the value 1 in x. Its deletion leaves a graph with a perfect 2-matching without odd polygons. By Lemma 2.1 this graph has a perfect 2-matching without any polygon and therefore it has a perfect matching. ▮

Lemma 3.5. *If G is a nonseparable hypomatchable graph, then there exists an ear decomposition of G such that, for every $i = 1, \dots, p$, the attachments of the ear P_i are distinct.*

Such an ear decomposition will be called a *nonseparable ear decomposition* of G.

Proof. Let x be a near-perfect matching of G. Construct an ear decomposition of G relative to x. If $G[V_i]$ is nonseparable for every $i = 0, 1, \dots, p$, then the lemma is proved. Othervise let i be the smallest index for which $G[V_i]$ is separable. Denote by w_i the cutnode of $G[V_i]$. We redefine the ear decomposition in the following way. For $j \geqq i$, if w_i is a cutnode of $G[V_j]$, then $V_j - V_{i-1}$ is adjacent to some vertex of $V - V_j$, say u_{j+1}, since G is nonseparable. Choose an ear P_{j+1} which starts with vertex u_{j+1}. (This is always possible by Remark 3.2.) Eventually there is an iteration $j > i$ where w_i is not a cutnode of $G[V_j]$. Consider the smallest such j. Let v_j be the endpoint of the ear P_j different from u_j. Two cases can occur.

Case 1. The vertex v_j is adjacent to some vertex $y \in V_{i-1} - \{w_i\}$. Then consider a vertex $z \in V_{j-1} - V_{i-1}$ adjacent to u_j. Since the graph $G[V_{j-1}]$ is hypomatchable and contains w_i as a cutnode separating $G[V_{i-1}]$ from the rest of $G[V_{j-1}]$, the graph $G[V_{j-1} - (V_{i-1} - \{w_i\})]$ must also be hypomatchable. Therefore, by Lemma 3.4, the graph contains an even length path from z to w_i whose removal leaves a graph with a perfect matching, say matching x'. This even path can be extended by an odd number of edges from z to y using the ear P_j. Let Q denote the resulting path from y to w_i. Next we will show that the portion Q_i of Q between v_j and the neighbor s of w_i can be used as a valid ear in iteration i of the ear decomposition.

Complete the matching x' to a near-perfect matching of $G \times V_{i-1}$ in the following way. Match every other vertex of Q, leaving s as a deficient node. Now x' can be completed so that every node of $G\times V_j$ is perfectly matched except the pseudonode, since $G\times V_j$ is hypomatchable.

Finally we modify the matching x in the following way. x stays unchanged in $G[V_{i-1}]$. In the rest of the graph, $x_e=x'_e$ except for the edges in the portion of the path Q which goes from y to s. For these edges $x_e=1-x'_e$. Since x was not changed in $G[V_{i-1}]$ the ear decomposition which was constructed relative to x is still valid up to the iteration $i-1$. Then, in iteration i, we can choose the vertex s as the start of the ear. (Remark 3.2.) Then using edges alternately in x and x' we follow the path Q. The condition (b) of Theorem 1.4 is satisfied. Furthermore the attachments w_i and y are distinct.

Case 2. The vertex v_j is not adjacent to $V_{i-1}-\{w_i\}$. Then define F as the set of all the edges of $G[V_j]$ which join a node of $V_{i-1}-\{w_i\}$ to a node of V_j-V_{j-1}. Since w_i is not a cutnode of $G[V_j]$ and j is the smallest index greater than i with this property, we know that F contains at least one edge, say edge (y, z). Now consider the graph G' obtained from $G[V_j]$ by removing the edges of F. Note that G' is hypomatchable, contains w_i as a cutnode separating $G[V_{i-1}]$ from the rest of G' and therefore the same argument as in Case 1 can be used to find the path Q, with the following changes: G' plays the role of $G[V_{i-1}]$ and the odd path from z to y is now reduced to the single edge (y, z). So, as in Case 1, we can construct an ear with distinct attachments w_i and y. □

Remark 3.6. At each step of the construction of a nonseparable ear decomposition, the choice of the attachment $w_i\in V_{i-1}$ is arbitrary as long as w_i is adjacent to $V-V_{i-1}$.

Lemma 3.7. *Let G be a nonseparable hypomatchable graph and let u and v be two distinct vertices of G. Then there exists an even path of length four or longer from u to v whose deletion leaves a graph with a perfect matching if and only if there does not exist a vertex w of G adjacent to only u and v.*

Proof. The "only if" part of the lemma is obvious since the deletion of a path of length 4 or more between u and v would leave the vertex w isolated.

Now we prove the converse. Consider the near-perfect matchings x^u and x^v deficient at u and v respectively. The removal of the path from u to v whose edges are assigned the value 1 in x^u+x^v leaves a graph with a perfect matching by Lemma 3.4. By assumption this path cannot have length 4 or more. So it has length 2. Let w be its middle vertex, and let x^w be a matching of G deficient at vertex w. Note that the even path from w to u whose edges are assigned the value 1 in x^u+x^w starts with the edge (w, v). By assigning the value 1 to the edge (u, w) we therefore obtain a perfect 2-matching of G whose only odd polygon contains the path (u, w, v). By Theorem 3.3 this odd polygon can be used as the first cycle of an ear decomposition. Let V_0 be the vertex set of this odd cycle C_0.

Suppose that w is adjacent to some vertex s distinct from u and v. If $s\in V_0$, then take the portion of the cycle from s to u or v, whichever has an even number of edges, say u for the sake of the argument. Then complete this path with the edges (s, w) and (w, v). This even path has length at least 4 and its removal leaves a graph with a perfect matching, a contradiction. So $s\in V-V_0$. Since G is nonseparable and hypomatchable we know by Lemma 3.5 that V_1 can be chosen so that $G[V_1]$ is nonseparable. In fact, by Remark 3.6, we can impose that w be an attachment of the ear P_1.

Let $y \neq w$ be the other attachment of the ear P_1. Consider the even path from u to v defined by: the section of the cycle C_0 from y to u or v whichever is even, say u; the odd path from y to w which follows the ear P_1; and finally the edge (w, v). The length of this path is at least 4 and its removal leaves a graph with a perfect matching, a contradiction. Therefore w can only be adjacent to u and v. □

4. Algorithm and Tutte type theorems

Given a graph $G=(V, E)$, a *P_k-matching* (not necessarily perfect) is a non-negative integral vector $x=(x_j: j\in E)$ satisfying $x(\delta(v))\leqq 2$ for all $v\in V$ and such that the edges assigned the value 1 do not form any polygon with k edges or less.

We now describe an algorithm which finds a P_k-matching x with the largest possible value $x(E)$. Of course, the problem P_k introduced in Section 2 will be solved as a by-product by checking whether $x(E)=|V|$. The algorithm is based upon an oracle which, given a hypomatchable graph, answers whether it is P_k-critical or not, and in the latter case exhibits a perfect P_k-matching. The implementation of such an oracle is discussed in Section 5.

In the course of the algorithm we grow an *alternating forest*. The nodes of the alternating forest F may be of two types. A *real node* of F is simply a node of G. A *shrunk node* of F is a vertex induced subgraph of G which is P_k-critical. The edges of F are edges of G where we consider an edge j to be incident with a shrunk node of F if exactly one end of j is in the shrunk node. Each tree in the forest is *rooted* at some node (which may be a real node or a shrunk node.) The nodes of F are designated as being *odd (even)* if the number of edges of F in the path to the root is odd (even). Odd nodes of F will always be real nodes.

An alternating forest F is always defined relative to a P_k-matching x (which will not be perfect) and must satisfy the following conditions. (See also Figure 2.)

(i) In every path in F from a root to another node of F the values x_j for the edges j in the path are alternately 0 and 2.
(ii) Each odd node of F is incident with two edges of F.
(iii) For every edge j which is not an edge of F but is incident with a node of F we have $x_j=0$.

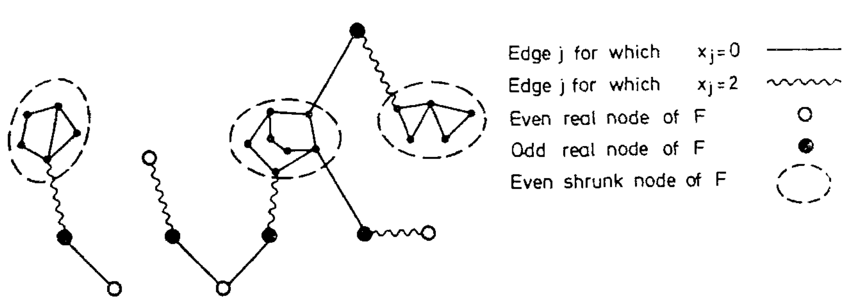

Fig. 2. Alternating forest

The algorithm starts with a (not necessarily perfect) P_k-matching x. It then attempts to "improve" x, if possible, in the following way. The nodes r such that $x_j=0$ for every j incident with r are the roots of an alternating forest. The forest is grown until either a means of augmenting the matching is discovered or no further growth is possible. In this latter case the algorithm discovers a structure which shows that $x(E)$ is as large as possible.

We now describe the algorithm in detail.

Step 0. (Initialization) Let x be any P_k-matching of G for which the edges assigned the value 1 form disjoint odd cycles. (For example $x_j=0$ for all $j\in E$.)

Step 1. (Optimality Test) If x is perfect, then terminate. Otherwise there are nodes r such that $x_j=0$ for every edge j incident with r. We now begin growing an alternating forest F whose roots consist of every such node r. Thus initially F consists of a set of even nodes.

Step 2. (Edge Selection) Find, if one exists, an edge j joining an even node u of F to a node v which is not an odd node of F. If no such node exists, terminate. Otherwise, 4 cases may occur.

Case 1. v is not a node of F and is incident with an edge k for which $x_k=2$. Go to Step 3 where we grow the forest.

Case 2. v is not a node of F and there are two edges h and l incident with v for which $x_k=x_l=1$. Go to Step 4 where we augment the matching.

Case 3. v is an even node of F and is in a different tree than u. Go to Step 5 where we augment the matching.

Case 4. v is an even node of F and is in the same tree as u. Go to Step 6 where we augment or shrink.

Step 3. (Forest Growth) Let w be the node incident with k which is different from v. Grow F by adjoining edges j and k and nodes v and w. Thus v becomes an odd node of F and w becomes an even node. Go to Step 2.

Step 4. (Cycle Breaking Augmentation) Edges h and l belong to an odd cycle P of G such that $x_k=1$ for every edge k in P. We now travel around P starting with h setting $x_k=0$ or 2 alternately for each edge k until we reach edge l. Then x_k and x_l will both be 0. Consequently every edge incident with v is assigned the value 0; consider v as a root of F and go to Step 5.

Step 5. (Simple Augmentation) Set $x_j=2$. Then traverse the path in F from u to its root, alternately lowering and raising by 2 the value x_h for each edge h of F encountered in this path. Perform the same operation on the path from v to its root. After this change any shrunk node K of F will have exactly one real node $w\in K$ incident with an edge k of F for which $x_k=2$. By Theorem 2.2, K is hypomatchable. Therefore the values of the edges of G inside the shrunk node can be modified so that w is the only deficient node. (This can be performed efficiently by adjoing the edge k to K and constructing an alternating tree relative to the old matching and rooted at the endpoint of k which is not in K. The first augmentation gives the required modification of x in K.) We now "throw away" F and any shrunk node previously formed and go to Step 1.

Step 6. (Augment or Shrink) Edge j added to F creates an odd cycle P. Let H be the subgraph of G induced by the real vertices of P and those inside shrunk nodes of P. Call the oracle to check the status of H. If H is P_k-critical, go to Step 6a. Otherwise go to Step 6b.

Step 6a. (Shrinking) Create a new shrunk node containing H. Now this shrunk node is an even node of F. Go to Step 2.

Step 6b. (Augmentation) The oracle exhibits a perfect P_k-matching of H. Complete the augmentation of x by alternately setting the edges to 0 or 2 in the path of F joining the odd cycle P to the root. If this path contains any shrunk node, the values inside these shrunk nodes can be modified as mentioned in Step 5. "Throw away" F and any shrunk node and go to Step 1.

End of Algorithm

Remarks on the algorithm. (i) The algorithm cannot cycle since at most $|V|$ augmentations can occur and, between augmentations, either the alternating tree is grown or an odd cycle is shrunk. Consequently the algorithm terminates in Step 1 or 2. The running time is bounded by $O(|V|^3)$, assuming that a call of the oracle takes unit time. Furthermore the number of calls of the oracle is at most $O(|V|^2)$.

(ii) To prove the validity of the algorithm, it suffices to show that, when the algorithm terminates in Step 2, no perfect P_k-matching exists and, in fact, the current solution x maximizes $x(E)$. Note that, at termination, every edge incident with an even node of F is also incident with an odd node of F. It follows that each even node of F is a connected component of $G[V-S]$ where S denotes the set of odd nodes of F. Moreover we know that all these even nodes are P_k-critical. Thus the value $x(E)$ cannot be improved since the vertex set S is already saturated by edges matched with the even nodes.

The validity of the algorithm provides a proof of the next theorem, and consequently of Theorem 2.5.

Theorem 4.1. *Let p be a nonnegative integer. A graph $G=(V, E)$ has a P_k-matching x with $x(E)=|V|-p$ if and only if, for every $S\subseteq V$, the number of P_k-critical components of $G[V-S]$ is not greater than $|S|+p$.*

Proof. If G has a P_k-matching x with $x(E)=|V|-p$, then clearly the number of critical components of $G[V-S]$ cannot exceed $|S|+p$ since $x(\delta(S))\leqq 2|S|$. This proves the necessity of the algorithm. The sufficiency follows from the validity of the algorithm. □

When $p=0$ this theorem reduces to Theorem 2.5. When furthermore $k=|V|$ we get Tutte's theorem (Theorem 1.2.). Similarly Theorems 2.3 and 2.4 are obtained when $k=2$ and 3 respectively. Note that when we set $k=|V|-1$ we get a theorem about Hamiltonian graphs with an odd number of vertices. Unfortunately the necessary and sufficient condition obtained in this case is a trivial condition for Hamiltonicity.

László Lovász [9] has observed that, in fact, the scope of Theorems 2.2, 3.1, 2.5 and 4.1 and the preceding algorithm can be extended in the following fashion. Let $\mathscr{C}$ be any set of polygons of G and let $P_{\mathscr{C}}$ denote the property "has a perfect 2-matching containing no polygons from $\mathscr{C}$". Then all these results and their proofs remain true with $P_{\mathscr{C}}$ substituted for P_k. Thus, although our work arose out of a desire to study 2-matchings that excluded "small" polygons, in fact it applies to arbitrary sets of excluded polygons.

5. Checking P_k-criticality

The algorithm presented in Section 4 relies on an oracle for recognizing whether a hypomatchable graph is P_k-critical. The purpose of this section is twofold. First we prove that the oracle can be implemented in polynomial time for any given k. Then we analyse the case $k=5$ in some depth and show how to generate the family of P_5-critical graphs constructively by specializing the ear decomposition of Lovász (Theorem 1.4).

Proposition 5.1. *Let $G=(V,E)$ be a hypomatchable graph and k an odd integer. G has a perfect P_k-matching if and only if there exists a path $\{v_1, e_1, v_2, \dots, e_k, v_{k+1}\}$ with $v_1 \neq v_2 \neq \dots \neq v_{k+1}$ such that $G[V-\{v_2, \dots, v_k\}]$ has two near-perfect matchings, one deficient at v_1 and the other deficient at v_{k+1}.*

Proof. Assume that there exists such a path and two near-perfect matchings x' and x'' deficient at v_1 and v_{k+1} respectively in $G[V-\{v_2, \dots, v_k\}]$. Adding up x' and x'' yields a 2-matching having no odd cycles. Any even cycle can be removed as described in the proof of lemma 2.1. The vertices v_1 and v_{k+1} are joined by an even path, all of whose internal vertices are unmatched by $x'+x''$. By assigning the value 1 also to the edges $e_1, e_2, \dots, e_k$ we get a perfect P_k-matching of G.

To prove the converse we use Theorem 3.1. If G has a perfect P_k-matching then it has one, say x, with exactly one odd polygon. Let $\{v_1, e_1, v_2, \dots, e_k, v_{k+1}\}$ be a portion of this odd polygon. Then in $G[V-\{v_2, \dots, v_{k+1}\}]$ the 2-matching x perfectly matches every vertex except v_1 and v_{k+1}. Furthermore the edges assigned the value 1 form an even length path joining v_1 to v_{k+1}. Therefore, in $G[V-\{v_2, \dots, v_k\}]$, x is the sum of two near-perfect matchings, one deficient at v_1 and the other deficient at v_{k+1}. □

As a consequence of Proposition 5.1, one can check the P_k-criticality of a hypomatchable graph G by enumerating all the paths of length k and, for each such path $(v_1, e_1, \dots, e_k, v_{k+1})$, by checking that $G[V-\{v_1, \dots, v_k\}]$ and $G[V-\{v_2, \dots, v_{k+1}\}]$ do not both have a perfect matching. (This assumes that k is odd. When k is even the solution follows from the property $P_k \equiv P_{k-1}$.) Of course such a simple-minded approach for checking P_k-criticality is not practical since the time required is of the order of n^{k+4}. By a better understanding of the structure of P_k-critical graphs we can hope for faster algorithms. It is easy to verify that a graph is P_k-critical if and only if each of its nonseparable components is P_k-critical. Furthermore, since every P_k-critical graph is hypomatchable, a natural approach is to specialize the ear decomposition of hypomatchable graphs in order to retain only those graphs that are P_k-critical and nonseparable.

We illustrate this approach when $k=5$ (or 6).

Theorem 5.2. *Let $G=(V, E)$ be a hypomatchable nonseparable graph and let $V_0, V_1, \ldots, V_p$ be any nonseparable ear decomposition of G. Then G is P_5-critical if and only if the following conditions are satisfied:*

(a′) *V_0 has cardinality 3 or 5.*

(b′) *For each $i=1, \ldots, p$, $V_i - V_{i-1}$ has cardinality 2, say $V_i - V_{i-1} = \{y, z\}$. Furthermore, for any pair of vertices $u \neq v$ in V_{i-1} such that u is adjacent to y and v is adjacent to z, there must exist a vertex $w \in V_{i-1}$ which is adjacent only to u and v.*

Proof. First we show that if G is P_5-critical and nonseparable, then any nonseparable ear decomposition satisfies (a′) and (b′).

Let V_i, $0 \leqq i \leqq p$, be a nonseparable ear decomposition of G. By Theorem 3.3 there exists a perfect 2-matching of G whose only polygon is the first cycle of the ear decomposition. By assumption on G, this 2-matching cannot be a perfect P_5-matching; therefore V_0 must have cardinality 3 or 5.

Now we consider an ear P_i. Let $u \neq v$ be attachments of P_i. (By Lemma 3.5 we know that at least one such pair u, v exists.) Since $G[V_{i-1}]$ is hypomatchable, there is an even length path from u to v whose removal leaves a graph with a perfect matching (Lemma 3.4). Since $G \times V_i$ is hypomatchable, a 2-matching of G can be found whose only odd cycle uses the even path from u to v and the ear P_i. Since G is P_5-critical, this odd cycle can have at most 5 edges. Therefore $|V_i - V_{i-1}| = 2$. Furthermore, by Lemma 3.7, $G[V_{i-1}]$ must have a vertex w which is only adjacent to u and v. So we have just proved that every nonseparable ear decomposition of a P_5-critical graph satisfies conditions (a′) and (b′).

Conversely suppose that the graph G has a nonseparable ear decomposition which satisfies (a′) and (b′). Clearly G is nonseparable. We will show that it is P_5-critical. By Theorem 1.4, G is hypomatchable. So it remains to prove that G does not have a perfect P_5-matching. We show this inductively by proving that $G[V_i]$ does not have one for $0 \leqq i \leqq p$. Clearly, by condition (a'), $G[V_0]$ does not have such a matching. Now assume that $G[V_{i-1}]$ does not have a perfect P_5-matching. Let $V_i - V_{i-1} = \{y, z\}$ and assume that $G[V_i]$ has a perfect P_5-matching, say x. We can assume that x has no even polygon. Note that $x_{yz} \neq 2$ since $G[V_{i-1}]$ does not have a perfect P_5-matching. If $x_{yz}=1$, then the edge yz belongs to an odd polygon of x which also goes through edges (u, y) and (v, z) where u and v both belong to V_{i-1}. Note that $u \neq v$ since no odd polygon of x is a triangle. In fact, since no pentagon is allowed either, the edges assigned the value 1 must form an even path of length four or more from u to v. But this leaves the vertex w defined in condition (b') as a deficient isolated node, a contradiction. So we must have $x_{yz}=0$.

Let G' be the graph obtained from $G[V_i]$ by removing the edge (y, z) (not used in the 2-matching x). Let S be the set of vertices of $G[V_{i-1}]$ adjacent to either y or z. In $G'[V_i - S]$ the number of isolated nodes is larger than $|S|$, since they inlude y, z and one vertex w for each pair of distinct vertices u, v where u is adjacent to y and v adjacent to z. (It is easy to check that the number of vertices w is at least $|S|-1$.) So by Theorem 2.3, G' does not have a perfect 2-matching. This contradicts the assumption about x. □

A consequence of Theorem 5.2 is that every P_5-critical graph with more than 5 vertices has at least one vertex with degree 2.

Theorem 5.2 can be used in Step 6 of the algorithm for checking P_5-criticality. Provided appropriate data structures are maintained, it is possible in time $O(|V|)$ to determine which of Step 6a or 6b should be performed. Moreover it appears that by carrying out an $O(|V|)$ update of the data structures in Step 6a it will be possible to perform any augmentation in time $O(|V|^2)$. Thus the algorithm will have an overall complexity of $O(|V|^3)$. However we have not verified the details.

6. Some Remarks about the Integer Polytopes

In this paper we have demonstrated that, in order to decide whether a graph $G=(V, E)$ contains a perfect P_k-matching, it is useful to introduce the concept of a P_k-critical graph. A more general problem is: given a real weight function w defined on the edges of G, find a P_k-matching x which maximizes the linear function $w \cdot x$. If $\mathscr{P}_k(G)$ denotes the convex hull of the P_k-matchings of G, the problem can be stated as $\max \{wx\colon x \in \mathscr{P}_k(G)\}$. We do not know an efficient algorithm for this problem when k is an integer between 5 and $|V|-1$. However when $k=1, 2, 3, 4$ or $|V|$ good algorithms exist and they all rely on a description of the convex hull of the feasible solutions by a system of linear inequalities. More precisely, in each of these 5 well-solved cases the essential inequalities are

$$x_e \geqq 0 \qquad \text{for all } e \in E,$$

$$x(\delta(v)) \leqq 2 \qquad \text{for all } v \in V,$$

$$x(\gamma(S)) \leqq |S|-1 \qquad \text{for all } S \subseteqq V \text{ such that } G[S] \text{ is nonseparable and } P_k\text{-critical.}$$

However this polyhedral description does not generalize to $5 \leqq k \leqq |V|-1$. In fact, some essential inequalities may have non $-0,1$ coefficients: consider the graph G and the set of coefficients a_e of Figure 3.

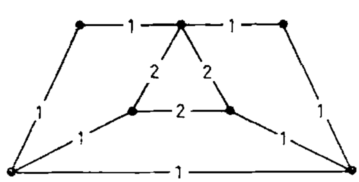

Fig. 3. Non 0,1 facet

The inequality $\sum_{e \in E} a_e x_e \leqq 8$ defines a facet of $\mathscr{P}_5(G)$, see [4, Theorem 3.2]. We conclude with some results concerning the 0,1 inequalities which are essential in the polyhedral description of $\mathscr{P}_k(G)$, i.e. the facet inducing inequalities of the form $x(F) \leqq f$, for $F \subseteqq E$.

Remark 6.1. *If* $G[S]$ *is nonseparable and* P_k*-critical, then* $x(\gamma(S)) \leqq |S|-1$ *induces a facet of* $\mathscr{P}_k(G)$.

Proof. The ineqality $x(S) \leqq |S|-1$ is valid for $\mathscr{P}_k(G)$ and it induces a facet of $\mathscr{P}_{|V|}(G)$ which is a full-dimensional polytope contained in $\mathscr{P}_k(G)$, see [12]. □

In the remainder we will consider subgraphs of G which are not necessarily node induced: let $G'=(V', E')$ where $V' \subseteqq V$ and $E' \subseteqq \gamma(V')$.

Remark 6.2. *Assume that $G'=(V', E')$ is a nonseparable P_k-critical subgraph of G and that it is edge maximal with this property. Then $x(E') \leqq |V'|-1$ induces a facet of $\mathscr{P}_k(G)$ if and only if*

(6.1) *for every $j \in \gamma(V')-E'$, the graph $G''=(V', E' \cup \{j\})$ has a perfect P_k-matching whose odd polygon contains the edge j.*

Proof. Assume that (6.1) is satisfied. The inequality $x(E') \leqq |V'|-1$ is valid. There are $|E'|$ affinely independent near-perfect $\{0, 2\}$-matchings of G' which satisfy $x(E')=|V'|-1$. Each edge $j \in E-\gamma(V')$ can be assigned the value 2 and completed into one of these near-perfect $\{0,2\}$-matchings. Finally, by the assumption (6.1), each edge $j \in \gamma(V')-E'$ can also be completed into a P_k-matching of G'' which satisfies $x(E')=|V'|-1$. It is easy to check that these $|E|$ P_k-matchings are affinely independent by using the fact that, for $j \in E-E'$, exactly one of them satisfies $x_j>0$.

Conversely if (6.1) is not satisfied, then there exists an edge $j \in \gamma(V')-E'$ such that every P_k-matching of G which satisfies $x(E')=|V'|-1$ also satisfies $x_j=0$. Since a facet is contained in a unique hyperplane, the inequality $x(E') \leqq |V'|-1$ does not induce a facet. □

Remark 6.3. *There exist edge maximal nonseparable P_k-critical subgraphs $G'= =(V', E')$, for which $x(E') \leqq |V'|-1$ does not induce a facet of $\mathscr{P}_k(G)$.*

Proof. Consider the graph G drawn in Figure 4.

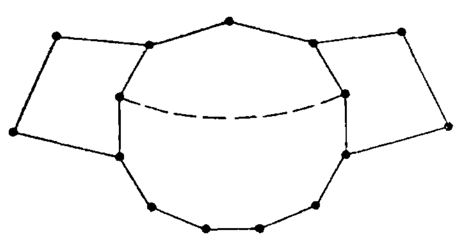

Fig. 4. Non facet-inducing subgraph

Let G' be the subgraph of G obtained by deleting the dotted edge. Note that G' is P_{11}-critical. Furthermore it is edge maximal for this property since G has a perfect P_{11}-matching. In fact, G has a unique perfect P_{11}-matching: it takes the value 2 on the dotted edge and the value 1 on the remaining 13-gon. Therefore the condition (6.1) of Remark 6.2 is not satisfied and consequently $x(E') \leqq |V'|-1$ does not induce a facet of $\mathscr{P}_{11}(G)$. □

Remark 6.4. *Assume $|V|$ odd and $k=|V|-2$. Then, for every edge maximal P_k-critical graph $G'=(V', E')$ the inequality $x(E') \leqq |V'|-1$ induces a facet of $\mathscr{P}_k(G)$.*

Proof. See [4]. □

Theorem 6.5. *For every edge maximal nonseparable P_5-critical graph $G'=(V', E')$, the inequality $x(E')\leqq|V'|-1$ induces a facet of $\mathcal{P}_5(G)$.*

Proof. Consider any nonseparable ear decomposition of G', which will satisfy the conditions (a′) and (b′) of Theorem 5.2. Let $j\in\gamma(V')-E'$. By applying the same ear decomposition to $G''=(V', E'\cup\{j\})$ there must be an iteration where the condition (b′) is violated, since G' is edge maximal P_5-critical. Let i be the smallest such iteration. We distinguish 2 cases depending on whether or not $j\in G''[V_{i-1}]$.

Case 1. $j\in G''[V_{i-1}]$. Then there must be attachments $u\neq v$ of the ear P_i and a vertex w which is only adjacent to u and v in G', but which is also incident with j in G''. Note that, in G'', no vertex is adjacent to only u and v. So, by Lemma 3.7, there is a path of length 4 or longer between u and v which can be completed into a perfect matching of $G''[V_{i-1}]$. In fact, following the proof of Lemma 3.7, it is possible to ensure that this path uses the edge j. Therefore a perfect P_5-matching whose odd polygon contains j can be found in $G''[V_i]$ and thus in G''. So (6.1) is satisfied.

Case 2. $j\notin G''[V_{i-1}]$. Let u be the endpoint of j which is in V_{i-1}. Since the condition (b′) of Theorem 5.2 is violated, there must be an attachment $v\neq u$ such that no vertex w of V_{i-1} is adjacent only to u and v. So again by Lemma 3.7 there is a path of length 4 of longer between u and v which can be extended into a perfect P_5-matching of $G''[V_i]$ whose odd polygon contains j. So (6.1) is satisfied again.

Therefore by Remark 6.2, the inequality $x(E')\leqq|V'|-1$ induces a facet of $\mathcal{P}_5(G)$. □

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